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IS INFLATION A STOCHASTIC PROCESS ?

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Abstract

We propose two models which attempt to account for the effect of fluctuations upon the dynamics of the scalar field during the process of inflation. We find that the effect can be dramatic. Large spatial fluctuations of the scalar field are dynamically generated. It is therefore unlikely that the inflationary process resembles the dynamical picture as proposed by the New Inflationary Scenario. We believe that the dynamics of the scalar field is more appropriately thought of as a process of "domain formation". In this respect, the scenario that we suggest is similar to the dynamical picture suggested by Mazenko, Unruh, and

Wald (MUW). That is, the dynamics of the symmetry breaking process will force the universe to fracture into domains of different degenerate vacua; bubble expansion is an inappropriate description of the process. However, our model contrasts with MUW's — we will show that the process of fracturing may require a substantial period of time, during which, the universe will inflate. Indeed, our model may be consistent with the New Inflationary Scenario if the expansion during the domain formation is sufficient to contain the observed universe within a single domain. It will be shown that due to the dynamical nature of the fluctuations, that it is likely that sufficient inflation may occur.

In both models that we shall present, the state of the universe is characterized by a probability distribution for the value of the scalar field throughout the universe. We thus avoid an implicit assumption of spatial homogeneity. For both cases, we assume for the sake of simplicity that measurements of the value of the value of the scalar field throughout the observed universe will be described by a Gaussian.

A simple static model is based upon an effective static energy, which is equal to the distribution averaged value of the scalar potential. From the form of this average potential, qualitative features of the inflation process are easily deduced. We find that it is likely that the universe first reduced its effective static energy by aggravating the scalar field fluctuations. Moreover, when the fluctuations are large enough, the evolution of $\langle\phi\rangle$ is delayed for a possibly long period of time at a value near zero. Since the energy of the universe is above the global minimum, it will inflate during this period of large fluctuations.

In a second model, the effect of quantum and thermal fluctuations upon the motion of the scalar field are simulated by adding a white noise term to a Linde–Albrecht–Steinhardt like equation. The result is a Langevin equation for the scalar field for inflation. In turn, this leads to a partial differential equation of motion for the phase space probability density, $W(\pi,\phi;t)$ which is recognized as a Fokker–Planck equation. We assume a Gaussian shape for the value of the scalar field distribution and then finally obtain a set of coupled ordinary differential equations for $\langle\phi\rangle, \langle\pi\rangle, \langle\phi^2\rangle, \langle\pi\phi\rangle$, and $\langle\pi^2\rangle$. These equations reduce to the usual LAS equation for $\langle\phi\rangle$ alone in the limit of a noise term of zero strength.

As a test of our dynamical model we calculate the time evolution in de Sitter space of (i) a free massive scalar field; (ii) a free massless scalar field; (iii) a massless scalar field under the influence of a $-\frac{\lambda}{4}\phi^4$ potential. For cases (i) and (ii) our results are in substantial agreement with previous calculations. By analogy with finite temperature quantum systems,we associate a temperature with steady state solutions of the Fokker–Planck equation. For the case of the free massive scalar field, we find that the temperature of the scalar field system is bounded above by $H/2\pi$, exactly. This is the same temperature that Gibbons and Hawking have associated with de Sitter space. In contrast, we find for the case of the unstable potential (iii), our results differ substantially from previous calculations. This discrepancy stems from the fact that in the earlier calculations, it is implicitly assumed that momentum fluctuations have no effect upon the dynamics of the scalar field.

Numerical solution of the stochastic equation of motion are found to support the qualitative results obtained from the static model.

1. INTRODUCTION

The New Inflationary Scenario (NIS) is an attractive model of cosmology for its clever resolution of the flatness–, horizon–, and monopole problems [1]. The NIS asserts that as the universe cools, the scalar field in isolated regions (bubbles or fluctuation regions) tunnel through a shallow potential barrier into an unstable portion of the scalar potential and then undergoes a uniform *rollover* down a very flat effective potential. During this slow roll down, the bubble is claimed to inflate to a size large enough to encompass the presently observed universe. The dynamics of a bubble are then described by the simple Linde–Albrecht–Steinhardt (LAS) equation,

$$\frac{d^2\langle\phi\rangle}{dt^2} = -3H\frac{d\langle\phi\rangle}{dt} - V'(\langle\phi\rangle) \tag{1.1}$$

However, it has been suggested recently that this beautiful picture is flawed. Knowledge of condensed matter models and quantum mechanical considerations indicate that it is unlikely that a bubble will evolve as a homogeneous system [2,3,6 — 8,13]. Since the LAS does not account for fluctuations, (1.1) cannot accurately reflect the dynamics of the process in which the scalar field changes its value from the symmetric state of $\phi \approx 0$ to the broken state where the value of the field is near one of the true minima of the scalar potential $\phi \to +\phi_c$.

We refer to this process in which the static energy of the region is minimized as *relaxation*. Bubble formation and evolution is an inappropriate picture of relaxation. We base this claim upon the following; (i) General arguments based upon physical reasoning indicate that during the cooling process the universe will not be supercooled to a state with $\phi \approx 0$, but, the field should form domains with ϕ near either $+\phi_c$ or $-\phi_c$ [2]. In other words, the picture of the slow homogeneous roll down of the scalar field is physically unlikely. (ii) The dynamics of the relaxation process will cause fluctuations of the scalar field to be enhanced. Even in the unlikely event of the formation of a homogeneous bubble, fluctuations in the value of the scalar field will render the LAS equation (1.1) to be an inappropriate description of the physics involved [3]. Further, if the fluctuations become so large (and we will argue that they will) that a bubble develops regions of $\phi \approx 0$ along with regions of $\phi = +\phi_c$ and $-\phi_c$, then the bubble becomes smeared with poorly defined edges. It becomes increasingly difficult (perhaps impossible) to characterize this volume as a single or many bubbles. We suggest that a common feature of many models is the energetic preference to enhance fluctuations, which in turn drives the system to break up into domains rather than evolve as a homogeneous bubble.

In this article we will present two simple models of domain formation. In both cases fluctuations play an important role. In Section 2 we put forth a simple model in which we calculate the effect of fluctuations (in the value of the scalar field) upon the energy of a static system. That is, we impose a static, non–homogeneous distribution of scalar field values upon the system and then calculate the resulting energy. In condensed matter physics this energy is sometimes called the *Coarse Grained Free Energy* [4]. This approach is in contrast with the usual calculation of the effective potential where it is arbitrarily assumed that the field is without fluctuations; i.e. $\langle\phi^n\rangle = \langle\phi\rangle^n$ for $n = 1, 2, \cdots$ [3]. We will assume by fiat that only $\langle\phi\rangle$ and $\langle\phi^2\rangle$ are independent and that the static energy of the system is a function of both. This approach has been suggested some time ago in the context of calculating effective potentials for composite operators [5] and it has been employed in an investigation of quantum fluctuations in the early universe [6]. From the form of this *effective static energy* for time independent distributions, we deduce features of relaxation (a non–static process). Our deductions will be based upon the premise that the dynamics of the relaxation process are impelled by the tendency to minimize the effective static energy of the system. We find that under a large class of conditions there is a strong dynamical tendency toward enhancement of the fluctuations of the scalar field. Even if the system is initially homogeneous, it will develop large fluctuations at some time during the relaxation process and will then eventually evolve to a time independent state that is locally homogeneous. On a global scale, the system will develop domains in which the scalar field has a constant value of either $+\phi_c$ or $-\phi_c$. Separating the domains are regions of transition, these are the walls in which the energy density is large. We claim that our models delineate the physics of formation of the *primordial domains* suggested by Mazenko, Unruh, and Wald (MUW) [2]. We refer to these domains as primordial, because they are the elements from which even larger domains are to be formed in a subsequent process of growth and coalescence. MUW claim that the observed universe will fracture itself into these primordial domains; in contrast, we suggest that the observed universe could ultimately be homogeneous, completely contained within one of these primordial domains.

Figures 1,2, and 3 depict a typical domain formation scenario as we envision it. Each figure is shaded such that: regions where the scalar field has a value $\phi \approx +\phi_c$ are white; regions where the scalar field has a value of $\phi \approx -\phi_c$ are black; and regions where the scalar field has a value of $\phi \approx 0$ are gray. The three figures represent distinct stages of the relaxation of the universe. In the center of each figure is marked a square which are implicitly divided into many smaller squares. (See Fig. 1 for and explicit demarcation of a smaller square.) In Section 2, the system which we model is represented by the large square in the middle of the figures . The very small sub–sections within the system are of importance in the dynamical model discussed in Section 3, but may be ignored in the context of Section 2. Figure 1. represents the universe in its initial moments where the scalar field is homogeneous throughout with $\phi \approx 0$, thus it is coated by a smooth shade of gray. Figure 2 represents a period of large fluctuation in the value of the scalar field. All areas of sufficient size within the frame are mottled with areas of white, black, and gray. The square in the middle of the figure is mottled, thus the distribution of scalar field values within the square has a significant width. On a much smaller scale (these sub–systems are marked explicitly in Fig. 1) the scalar field is homogeneous always. In Fig. 3, the universe has settled into a time independent state of clearly demarcated domains of either black or white. Any regions of gray that remain are contained in the inter–domain walls. These intermediate regions represent an insignificant fraction of the total volume of the universe. In theories of more complicated group structure, topological defects may arise near the domain intersections. We shall assume that their density is diluted à la inflation. Non–local objects such as cosmic strings will not be considered here. As depicted in Fig. 3, in the final stage of the relaxation process, the system (i.e. the large box) is completely contained within a domain, fluctuations have become insignificant, and the system is ultimately homogeneous. In our view, the large square shown within Figures 1, 2, and 3 represents the observed universe. In the end, the observed universe is completely contained within one of the primordial domains.

In order for a primordial domain as seen in Fig. 3, to contain the observed universe, the system must have inflated to a sufficient size during the relaxation process. MUW argue that the primordial domain formation occurs "quickly", thus prematurely ending the inflation process. However, we find that under a large class of conditions, the process of relaxation into domains occurs over a long period of time. The development of the scalar field distribution is extremely slow during the period of large fluctuations. We believe that such a slowing in the evolution of the distribution occurs because the effective static potential has a flat spot at a point where the average value of the scalar field within the system is small while the fluctuations are large. The reader should not confuse this "flatness" of the effective static potential with the small slope of the Coleman–Weinberg effective potential. The origins are completely different. Moreover, the large fluctuations in the value of the scalar field causes the system to have an average potential energy substantially above the minimum of zero. This means it might be possible that significant inflation occurs during the formation of the primordial domains. It is our contention that during the formation of the primordial domains suggested by MUW, there was an extended period of large fluctuation in which sufficient inflation could occur. Loosely speaking, we agree with MUW that domains are formed, but we disagree as to when inflation occurs.

We emphasize here that the conclusions we reach concerning the development of the scalar field will not depend upon the initial configuration of the scalar field. The starting configuration will not have a substantial effect upon the inflation process. It is, however, crucial that the universe spend a large amount of time in the flat spot of the effective static energy potential. The ensuing period of large fluctuations will effectively negate any effect of the initial condition of the scalar fields. Indeed, if the scalar field does not start out homogeneous as depicted in Fig. 1, but rather starts out in a state of large fluctuations, depicted in Fig. 2, the conclusions reached here about the stochastic nature of the inflation process will be unchanged.

The static model of relaxation presented in Section 2 is appealing because it is simple. However, the model is not entirely consistent. The final state of the system is a set of domains in each of which the scalar field is completely homogeneous. This absolute homogeneity of the system is in contradiction with the knowledge that the system will be at a small but finite temperature. The system is subject to both quantum and thermal fluctuations which will make the probability of observing a completely smooth system very remote.

The starting point for a dynamical model of the relaxation process is an empirically derived equation of motion for the scalar field in a microscopic subsection of the system. These sub–systems are assumed so small that the scalar field is assumed to be homogeneous always (see for example the small divisions of the system as drawn in Fig. 1). The assumption of homogeneity would normally yield an equation of motion for the sub–system that is identical to the LAS equation (1.1). However, each of the subsystems is coupled with external fields (eg. gravity) and other sub–systems. Energy will be exchanged in and out of a given system in a more or less random manner. On the scale of the whole system, such processes are identified as fluctuations. To model this fluctuation process, the equation of motion for the scalar field of the sub–system is modified to include a stochastic force term $\eta(t)$. Each of the sub–systems that make up the system obeys a unique differential equation of motion, and so fluctuations within the system ensue. The resulting stochastic differential equation for the sub–system is a Langevin equation. The Langevin equation is then converted into a partial differential equation of motion for the probability density in the phase space of the scalar field for the system. This is a standard Fokker–Planck or Kramers equation. We then assume that the phase space probability is approximately Gaussian in the ϕ–direction. As a result, the partial differential equation is converted to a system of five coupled non–linear ordinary differential equations. The dynamical variables in these differential equations are time dependent expectation values of various powers of the scalar field ϕ, and its conjugate momenta $\pi = d\phi/dt$. We then analyze the stationary states of the set of ODE's. It is found that the time development of the phase space of the system will slow at a point of large fluctuation and then come to rest at a point of lowest system energy (i.e. $\phi \approx \pm\phi_c$). The position of this slowing point is in quantitative agreement with the position of the *flat spot* of the averaged potential found in Section 2. In the final time independent state found as $t \to \infty$, the system is not completely homogeneous, but has a remnant fluctuation in both ϕ and π.

This remnant fluctuation is interpreted as an effect of finite temperature quantum statistical mechanics. An explicit and quantitative analysis of this remnant phase–space "noise" is given in the Appendix. By comparing the Langevin expectation values with an analogous quantum system, the strength of the stochastic noise term in the Langevin equation can be calculated up to an unknown length scale.

In Section 4 we test our dynamical formalism developed in Section 3 by considering several special cases which have been considered previously by other authors. We develop the Fokker–Planck equation associated with a free, massive, scalar field in a de Sitter space. The equation of motion for $\langle\phi^2\rangle$ is found to be almost identical to the analogous equations of motion obtained by looking at the evolution of the quantum modes in de Sitter space [7]. In particular, for $m^2 > 0$ the scalar field is found to evolve monotonically toward a final stable state in which $\langle\phi^2\rangle$ achieves a fixed nonzero value. This remnant value of $\langle\phi^2\rangle$ is a function of the parameters of the model. Bunch and Davies [8] have calculated the value of the square of the scalar field in such a de Sitter invariant vacuum to be $\langle\phi^2\rangle = 3H^4/8\pi^2m^2$. If we adjust the parameters of the theory to yield $\langle\phi^2\rangle \to \langle\phi^2\rangle_{BD}$, we find that the characteristic temperature of the scalar field system to be bound above by the de Sitter space Hawking temperature, $T_H = H/2\pi$.

For the massless, non–interacting scalar field, the vacuum is found to be unstable. This instability manifests itself by the result that the value of $\langle\phi^2\rangle$ diverges linearly with time. The coefficient of the linear divergence is found to be the same as found previously [7].

For small values of the scalar field $|\phi| \approx 0$, the Coleman–Weinberg effective potential [9] is approximated by $\Gamma[\phi] \approx -\lambda\phi^4/4$. In applying the dynamical scheme of Section 3 we discover equations of motion that differ significantly form the equations of motion derived by Vilenkin [7]. We can regain Vilenkin's equations if we impose the unnatural condition that the momentum distribution in the phase space of the scalar field is unchanged, remaining at its equilibrium value throughout its evolution. For the general case where the value of ϕ is centered about an unstable value of the potential, we suggest that our equation of motion will be in disagreement with those of other authors. The discord between our's and previous formalisms arises because we treat the momentum distribution in the phase–space of the scalar field to be a dynamical variable. The shape of the probability distribution will take on a constant value characteristic of thermal equilibrium only at the last stages of the relaxation process. In our view, the concepts of thermal equilibrium and temperature are useful only for slowly evolving states under the influence of a stable potential. This is certainly not the case in the early stages of the inflation process.

Section 5 presents the results of numerical solutions of the coupled ordinary differential equations developed in the dynamical model of Section 3. It is found that the relaxation process may follow two qualitatively different scenarios depending upon the values of the parameters of the model and the initial conditions imposed. In some cases, the relaxation process is similar to the dynamics of the bubble relaxation in the NIS. That is, the fluctuations of the scalar field do not grow excessively large and there is no evidence of the fluctuation induced slowing of the relaxation. In another larger class of relaxation scenarios, there is observed a period of large fluctuations in the value of the scalar field and the predicted slowing of the relaxation is seen. The value of the scalar field and its spread σ, is in agreement with the predictions of the static model of Section 2 and the steady state solutions of the dynamical equations of Section 3.

In the Conclusion we will give a terse review of our view of the early stages of the universe and how it differs from earlier pictures. We will indicate the areas where are arguments are weak and give indications of directions of research that we feel will be fruitful to pursue.

2. A Static Model

It is unlikely any physical system can be completely homogeneous. Even in the most ordered of states, fluctuations (of quantum and thermal origin) will result in a non–uniform distribution of values for any field, $\phi(x)$. We define the resulting field distribution by

$$W(\phi,t)d\phi = \text{probability that a measurement of } \phi(\vec{x},t) \text{ at time } t \text{ will lie in the range } (\phi, \phi + d\phi), \text{ where } \vec{x} \text{ is an arbitrary point within the volume of the system}$$

In this section we consider a simple static model in which we will calculate the energy dependence of a system upon a time independent probability distribution, $W(\phi)$. Knowledge of this static model will allow us to deduce general qualitative features of relaxation processes. In a later section, we will develop a dynamical model of relaxation in which $W(\pi = d\phi/dt, \phi; t)$ is the fundamental dynamical variable. The results of calculations in the dynamical model, will support conclusions reached here.

It is important to note that the correct way to calculate the steady state properties of a system is to employ well known effective potential techniques [9]. However, we will show that once one strays away from the steady state case, many conclusions about dynamic processes obtained from considering only static effective potentials can be misleading.

If the fluctuations in the field ϕ are of long enough wavelength, the average static energy density of the system will be due entirely to the potential energy term of the action;

$$U = \int d\phi W(\phi) V(\phi) \ . \tag{2.1}$$

It turns out that in de Sitter space, the long wavelength approximation is especially good because the corresponding d'Alembertian weights the ∇^2 term by a factor of e^{-3Ht}. Because we have neglected the kinetic energy term in the action the energy associated with domain walls are ignored. The system which we model is then, by fiat, a subset of a single domain. In our model of relaxation, a domain becomes well defined only in the later stages of the process. As we shall see, in the intermediate stages of relaxation, all regions have field values with large fluctuations. It is therefore difficult to separate one region from another. In the later stages of the relaxation process, the field value fluctuations subside and then regions that are homogeneous and clearly defined finally appear. Regions which are separated by distances much greater than some maximum correlation length of the theory (eg. a horizon) can evolve to different ground states, and domain walls will have formed to separate them. In some sense, our model is a microscopic description of the microscopic domain formation suggested by Mazenko, Unruh, and Wald [2], however our interpretation will be very different. If we consider the kinetic energy term significant only near domain walls, our neglection of this term in the energy of the system is justified. Of course, such bold assumptions should be checked later in a more sophisticated model.

Consider a system characterized by a single component, real field, ϕ. We assume the simplest form of spontaneously broken potential, namely a quartic potential with a $\phi \to -\phi$ symmetry:

$$V(\phi) = -\frac{\gamma}{2}\phi^2 + \frac{g}{4}\phi^4 \ . \tag{2.2}$$

We also assume that the distribution of values for the field is well approximated by a Gaussian

$$W(\underline{\phi}, \langle\underline{\phi}\rangle, \underline{\sigma}) = \frac{1}{\sqrt{2\pi\underline{\sigma}}} \exp\left(\frac{-(\underline{\phi} - \langle\underline{\phi}\rangle)^2}{2\underline{\sigma}}\right) , \tag{2.3}$$

where $\underline{\phi} = \sqrt{g/\gamma}\ \phi$ and $\underline{\sigma}$ are dimensionless. The average static energy density of the system is then given by the Gaussian weighted average of the quartic potential

$$\begin{aligned} \underline{U}(\langle\underline{\phi}\rangle, \underline{\sigma}) &= \frac{g}{\gamma^2} \int d\underline{\phi} W(\underline{\phi}; \langle\underline{\phi}\rangle, \underline{\sigma}) V(\phi) \\ &= -\frac{\langle\underline{\phi}\rangle^2}{2} + \frac{\langle\underline{\phi}\rangle^4}{4} + \frac{\underline{\sigma}(3\underline{\sigma} - 2 + 6\langle\underline{\phi}\rangle^2)}{4} \ . \end{aligned} \tag{2.4}$$

This dimensionless two parameter potential is shown in Fig. 4 . In our Gaussian approximation the distribution at any time is specified by a single point in the two dimensional $\langle\underline{\phi}\rangle$–$\underline{\sigma}$ space. The development of the distribution in a full dynamical model is driven by the tendency to minimize in a continuous fashion the energy of the system. The evolution of a dynamic distribution is isomorphic to a ball rolling down this two dimensional surface. Of course, if the ball is to ever come to rest (i.e. the distribution achieve a steady state) there must be some form of energy loss mechanism. Reason dictates that the energy loss should be a function of the velocity of the ball (i.e. $d\phi/dt$). To lowest order such an energy loss mechanism has the form of a simple frictional force resisting motion in the $\langle\underline{\phi}\rangle$–direction. Therefore, the ball and hill should be thought of as being immersed in a nonisotropic viscous medium and the motion of the ball is modified accordingly. We will not address the microscopic nature of this dissipative mechanism, but we will take its effect into account by the inclusion of an empirical friction term in the dynamical equations developed in the next section. With our static model now defined, it is now easy to deduce many of the important features generic to the relaxation process.

In Fig. 4, two qualitatively different paths of motion (on the two dimensional energy density surface) are shown. Path A, follows closely the line $\underline{\sigma} = 0$ from the symmetric $\langle\underline{\phi}\rangle = 0$ state to the broken state, $\langle\underline{\phi}\rangle = +1$. The equations of motion associated with such a path are analogous to the equations of motion proposed for the scalar field in the NIS (with the modification that a Coleman–Weinberg potential replaces the simple quartic potential used here). In the NIS, fluctuations are ignored ($\underline{\sigma} = 0$) and thus are not accounted for in the equations of motion. However, a path such as A is unlikely in our model. The tendency to minimize the energy of the system will enhance the fluctuations of the distribution (during the initial stages of the relaxation process) thus driving the ball away from the $\underline{\sigma} = 0$ line. Fluctuation enhancement should be a feature common to most theories. Since the distribution, $W(\underline{\phi})$, is spread in $\underline{\phi}$–space, the field will experience different forces at different spatial points. As a result, the distribution is stressed and will spread with time if it is centered about a value of ϕ where $\partial^2 V/\partial\phi^2 < 0$ (this is indeed the case for the initial stages of relaxation). The upshot is that our model cannot sustain a well defined bubble. The dynamics favors periods of large fluctuations in the field value which will smear out the bubble. Since we feel that the enhancement of fluctuation should be a feature general to most models of inflation, we must reject the NIS model of bubble creation and evolution.

Our model gives results that are closer in nature to a picture of symmetry breaking presented by Mazenko, Unruh, and Wald (MUW). That is, the universe will not form isolated bubbles that form and evolve, rather, the universe will tend to evolve in a continuous manner from a homogeneous symmetric state to a state where the universe has broken into well defined domains of different degenerate vacua. Our picture departs from that of MUW because we claim that the period of inflation could occur during the evolution from the homogeneous to the domain dominated state. MUW rejected such a picture because they argue that this process occurs very quickly.

It is possible that the domain formation phase of the evolution of the universe could be sufficiently long as to allow inflation of the observed universe. The shape of the potential gives rise to another feature that should be common to most relaxation processes. Fig. 4 reveals a flat part in the potential located along the $\langle \underline{\phi} \rangle = 0$ line at a nonzero value of $\underline{\sigma}$. Indeed, the averaged quartic potential (2.5) has a saddle point at $(\langle \underline{\phi} \rangle = 0,\ \underline{\sigma} = 1/3)$. If the ball passed through this region its motion would slow down (see, for example, path B in Fig. 4). A measurement of the distribution as a function of time would show that there would be a dramatic slowing in its rate of development — simultaneously there is significant enhancement of the fluctuations, and the system energy is seen to be above the global minimum. This feature should be common to all spontaneously broken potentials averaged by a Gaussian probability distribution.

In this model the stress energy tensor does not immediately vanish as suggested by MUW, but remains nonzero for a significant period thus continuing the inflation process. This feature could well be a spurious byproduct of the Gaussian shape approximation. On the other hand, if our model is a correct reflection of the physics of the relaxation process, then the New Inflationary Scenario could still be valid in the presence of domain formation with the appropriate modification of the picture of the evolution of the scalar field.

This static model illustrates most clearly the important new qualitative physics that arise upon the inclusion of fluctuations into the dynamical picture. The generalization of this model is straight forward, but the general features of fluctuation enhancement and slowing down should remain.

III. A Dynamical Model of Relaxation

The dynamical system of the NIS is the famous bubble — an isolated region surrounded by an environment in which the scalar field remains trapped in a symmetric state. LAS make the bold assumption that the bubble remains uniform during the slow roll down of the scalar field down the flat potential. Our model is very different, in fact bubbles never exist. Under the process of relaxation, the entire universe undergoes the transformation from an initial uniform state where $\phi \approx 0$ everywhere, to a state where the universe is divided up into clearly defined primordial domains of either ground states, $\phi \approx \pm\phi_c$. A similar state has been suggested by MUW, but their interpretation differs crucially from ours.

The dynamical system of the Section 2 was a section of the universe which evolved to become the observed universe, which itself is a subset of a primordial domain. During the relaxation process, the observed universe experienced moments in which the scalar field was not homogeneous. However, on a much smaller scale than the size of the observed universe, the scalar field will always be homogeneous. It is such a sub–system of the observed universe that will be the focus of this section. Small sub–systems are depicted in Fig. 1. The global behavior of the non–homogeneous macro–system (i.e. the observed universe) is controlled by the collective behavior of the constituent sub–systems.

The study of the dynamics of relaxation processes is the subject of a large body of scientific literature. For the most part, equations of motion have been derived on empirical foundations. However, there has been recent progress in the development in the theory of dissipative quantum systems. Caldeira & Legget and Decker [10] have shown that to leading order in $\hbar$, the equation of motion for the density matrix operator (the Wigner distribution) is recognized as a Fokker–Planck or Kramers equation. Inspired by these results, we will use empirical arguments to develop a Fokker–Planck equation for the scalar field. The connection between the microscopic field theory and the Fokker–Planck equations shall be deferred to future investigations.

The simplest extension of the static potential (2.4) to a dynamical model is to postulate a Hamiltonian for the system of the form:

$$H(\pi, \phi) = \frac{\pi^2}{2} - \frac{\gamma}{2}\phi^2 + \frac{g}{4}\phi^4 , \tag{3.1}$$

where $\pi = d\phi/dt$. As this stands, the resulting equations of motion would be flawed. The energy of the system is conserved if its motion is described by (3.1). Energy conservation is in contradiction with the notion of relaxation from an energetically excited state to a state of lower energy. A second flaw inherent with (3.1) is that the equations of motion are deterministic. That is, all systems and all subsections of the system (sub-systems) will have the same value of the field, assuming that the initial conditions are the same. Fluctuations are excluded by deterministic equations of motion. If we regard the observed universe (the system represented as boxes in the middle of Figs. 1, 2, and 3) as composed of an ensemble of homogeneous sub-systems (see the small squares within the system in Fig. 1) then an appropriate equation of motion for a member of the ensemble is:

$$\frac{d^2}{dt^2}\phi = -\beta\frac{d}{dt}\phi + \frac{\gamma}{2}\phi^2 - \frac{g}{4}\phi^4 + \eta(t) , \tag{3.2}$$

where ϕ is the value of the scalar field within the small homogeneous sub-system. To take into account the fact that thermal and quantum effects will cause the ensemble of sub-systems to be non-uniform, we have added a random external force term $\eta(t)$. This force term can be thought of as modeling the energy exchange between various sub-systems and with external fields. The behavior of the system as a whole will be the result of the collective motion of all the sub-systems which make it up. A frictional energy loss term $\beta(d\phi/dt)$, has been included to assure that relaxation can occur. In the context of inflationary models, if we assume the evolution is de Sitter phase dominated, then $\beta = 3H$ and the energy loss is due to adiabatic cooling. A quantitative justification for equations of motion such as (3.2) is given in Section 4.

The equation of motion for the scalar field (3.1) differs from the corresponding equation suggested by Linde, Albrecht, and Steinhardt [1]. In our model, we employ a simple quartic potential for the energy term. Radiative corrections will certainly alter the form of this potential and so it would have been more correct to substitute the derivative of an appropriate effective potential, $\Gamma[\phi]$. However, the qualitative nature of the relaxation process should not be sensitive to the nature of the effective potential. Thus, for the sake of clarity we have chosen the naive quartic at the possible expense of physical relevance. Another distinguishing feature of our model is the inclusion of a stochastic force term, $\eta(t)$, into the equation of motion for the scalar field. Linde, Albrecht, and Steinhardt assumed the entire bubble (that is to become the observed universe) to be completely homogeneous at all times. In contrast, we have divided the system (the observed universe) into sub–systems which are small enough to have homogeneous field values. However, each sub–system may have different values of the field thus making the system heterogeneous, i.e. having a distribution for the ensemble of values for the field. Fluctuation of quantum and thermal origin will cause the distribution to have a nonzero width (to lowest order in $\hbar$ it has been shown that the effects of quantum and thermal fluctuations are indistinguishable in the Fokker–Planck equation [10]). The cause of the fluctuations between sub–systems is modeled in the equation of motion for the scalar field by the inclusion of the stochastic force term $\eta(t)$.

Equation (3.2) for the scalar field within a sub–system is a Langevin equation. To avoid confusion, we emphasize here that (3.1) is not to be identified with the Langevin equation of the *Stochastic Quantization Scheme* [11]. In Stochastic Quantization, the time variable is an extra time, a fictitious fifth dimension. Here, the time is real and the dynamics have physical meaning.

Defining the following dimensionless variables (identified by underscores):

$$\underline{t} \equiv \beta t \ ,$$

$$\underline{\phi} \equiv \sqrt{\frac{g}{\gamma}}\phi \ ,$$

$$\pi \equiv \frac{d\phi}{dt} \equiv \beta\sqrt{\frac{\gamma}{g}}\frac{d\underline{\phi}}{d\underline{t}} \equiv \beta\sqrt{\frac{\gamma}{g}}\underline{\pi} \ ,$$

$$\underline{\alpha} \equiv \frac{\gamma}{\beta^2} \ ,$$

$$\eta(t) \equiv \beta^2\sqrt{\frac{\gamma}{g}}\underline{\eta}(\underline{t}) \ , \tag{3.3}$$

the resulting dimensionless Langevin equation is:

$$\frac{d^2\underline{\phi}}{d\underline{t}^2} + \frac{d\underline{\phi}}{d\underline{t}} - \underline{\alpha}\underline{\phi} = \underline{\eta}(\underline{t}) - \underline{\alpha}\underline{\phi}^3 \ . \tag{3.4}$$

Where $\underline{\pi}$ is the dimensionless conjugate momentum to the dimensionless field variable $\underline{\phi}$. The Langevin equation (3.4) is conventionally expressed as two first order differential equations, namely

$$\frac{d\underline{\phi}}{d\underline{t}} = \underline{\pi} \ , \tag{3.5}$$

$$\frac{d\underline{\pi}}{d\underline{t}} = -\underline{\pi} + \underline{\alpha}\underline{\phi}(1 - \underline{\phi}^2) + \underline{\eta}(\underline{t}) \ . \tag{3.6}$$

This dimensionless Langevin equation is easily converted to an equation of motion for $W(\underline{\pi}, \underline{\phi}; \underline{t})$,the probability density in the canonical phase space of the ensemble [12]. Note that

$W(\underline{\pi}, \underline{\phi}; \underline{t}) d\underline{\pi} d\underline{\phi} =$ probability of finding at time $\underline{t}$,
a particular sub–system to be in an
infinitesimal volume element of phase space
$(d\underline{\pi} d\underline{\phi})$ near the point $(\underline{\pi}, \underline{\phi})$

The resulting equation is the Fokker–Planck or Kramers equation for the system,

$$\frac{\partial W}{\partial \underline{t}} = -\frac{\partial}{\partial \underline{\phi}}(\underline{\pi} W) + \frac{\partial}{\partial \underline{\pi}}\Big[\big(\underline{\pi} - \underline{\alpha}(1 - \underline{\phi}^2)\big) W\Big] + \underline{\epsilon}\frac{\partial^2}{\partial \underline{\pi}^2} W \ , \tag{3.7}$$

where we have assumed the stochastic noise term has the following ensemble average properties:

$$\langle \underline{\eta}(\underline{t}) \rangle = 0 \ , \tag{3.8}$$

$$\langle \underline{\eta}(\underline{t}_1) \underline{\eta}(\underline{t}_2) \rangle = 2\underline{\epsilon}\delta(\underline{t}_1 - \underline{t}_2) \ . \tag{3.9}$$

It should be mentioned that Guth and Pi have employed $W(\pi, \phi; t)$ to analyze fluctuations of the scalar field for some idealized potentials [13]. However, they impose rather severe constraints upon the shape of the potential in the π–direction. We define the ensemble average of any function of the phase space variables $\underline{\pi}$ and $\underline{\phi}$ as:

$$\langle \underline{F}(\underline{t}) \rangle \equiv \int d\underline{\pi} d\underline{\phi} F(\underline{\pi}, \underline{\phi}) W(\underline{\pi}, \underline{\phi}; \underline{t}) \ . \tag{3.10}$$

The Fokker–Planck equation (3.7) yields equations of motion for the ensemble average of $F(\underline{\pi}, \underline{\phi})$ by multiplying it by $d\underline{\pi} d\underline{\phi}$ and then integrating over phase space. With no other assumptions, the equations of motion for any such function will couple the lower powers of $\underline{\pi}$ and $\underline{\phi}$ to higher powers resulting in an infinite tower of coupled ordinary differential equations. However, if we impose the following mean field assignments to the expectation values of ϕ:

$$\langle\phi^3\rangle = \langle\phi\rangle\Big(3\langle\phi^2\rangle - 2\langle\phi\rangle^2\Big) \; ,$$
$$\langle\phi^4\rangle = 3\langle\phi^2\rangle^2 - 2\langle\phi\rangle^4 \; , \tag{3.11}$$

we obtain a closed set of coupled nonlinear ordinary differential equations. The mean field assignment (3.11) is sometimes called the Hartree–Fock approximation, but we view it as approximating the probability distribution in the ϕ–direction of phase space by a Gaussian. The result is a set of five first order nonlinear differential equations which couple the motion of $\langle\underline{\phi}\rangle, \langle\underline{\pi}\rangle, \langle\underline{\phi}^2\rangle, \langle\underline{\pi}^2\rangle$, and $\langle\underline{\pi}\underline{\phi}\rangle$:

$$\frac{d\langle\underline{\phi}\rangle}{d\underline{t}} = \langle\underline{\pi}\rangle \; ,$$
$$\frac{d\langle\underline{\pi}\rangle}{d\underline{t}} = -\langle\underline{\pi}\rangle + \underline{\alpha}\langle\underline{\phi}\rangle\Big(1 - 3\langle\underline{\phi}^2\rangle + 2\langle\underline{\phi}\rangle^2\Big) \; ,$$
$$\frac{d\langle\underline{\phi}^2\rangle}{d\underline{t}} = 2\langle\underline{\pi}\underline{\phi}\rangle \; ,$$
$$\frac{d\langle\underline{\pi}^2\rangle}{d\underline{t}} = -2\langle\underline{\pi}^2\rangle + 2\underline{\epsilon} + 2\underline{\alpha}\Big(\langle\underline{\pi}\underline{\phi}\rangle\Big(1 - 3\langle\underline{\phi}^2\rangle\Big) + 2\langle\underline{\pi}\rangle\langle\underline{\phi}\rangle^3\Big) \; ,$$
$$\frac{d\langle\underline{\pi}\underline{\phi}\rangle}{d\underline{t}} = \langle\underline{\pi}^2\rangle - \langle\underline{\pi}\underline{\phi}\rangle + \underline{\alpha}\Big(\langle\underline{\phi}^2\rangle\Big(1 - 3\langle\underline{\phi}^2\rangle\Big) + 2\langle\underline{\phi}\rangle^4\Big) \; . \tag{3.12}$$

To investigate the possible steady state solutions we set the time derivatives in (3.12) to zero. It is trivial to show that all of the steady state solutions satisfy:

$$0 = \langle\underline{\pi}\rangle = \langle\underline{\pi}\underline{\phi}\rangle \; ,$$
$$\langle\underline{\pi}^2\rangle = \underline{\epsilon} \; , \tag{3.13}$$

while $\langle\underline{\phi}\rangle^2$ and $\langle\underline{\phi}^2\rangle$ have several alternative solutions which are listed in Table 1, where $\underline{\sigma} \equiv (\langle\underline{\phi}^2\rangle - \langle\underline{\phi}\rangle^2)$ is the variance of the assumed Gaussian distribution

($\underline{\sigma}$ is the same quantity introduced in equation (2.4) and plotted in Fig. 2). In Table 1 the solution labeled (i) has (in the limit $\underline{\epsilon}/\underline{\alpha} \rightarrow 0$): $\langle\underline{\phi}\rangle \approx 1$ and $\underline{\sigma} \approx \underline{\epsilon}/2\underline{\alpha}$. This corresponds to the energy minimum solution of the Gaussian averaged potential $\underline{U}$ (2.4). The remnant fluctuations of the scalar field distinguishes the dynamical theory from the simple static model presented in the second section (where the final state has $\underline{\sigma} = 0$). The time independent primordial domains are not completely homogeneous but will have a remnant fluctuation of $\underline{\sigma} = \underline{\epsilon}/(2\underline{\alpha})$ and a quantum/thermal motion of $\langle\underline{\pi}^2\rangle = \underline{\epsilon}$.

If the Langevin equation (3.12) is a reasonable model of the physics, then the final state of the system should be the same state as predicted by finite temperature quantum mechanics. The final state of the system is centered about the minimum of the potential ($\phi = \phi_c$) and if we assume that the energy of the system is is low enough, then the final state physics of the system is well approximated by a harmonic oscillator. If we set the quantum expectation values of $\underline{\sigma}$ or $\underline{\pi}^2$ for the appropriate harmonic oscillator at finite temperature to the corresponding quantities of the Fokker–Planck equation (listed in (i) of Table 1.) , then we are lead to:

$$\frac{\underline{\epsilon}}{2\underline{\alpha}} = \frac{g}{(\omega L)^3}\coth\left(\frac{\omega}{2T}\right) , \tag{3.14}$$

where L is the spatial size of a typical homogeneous sub–system and,

$$\omega \equiv \sqrt{2\gamma} , \tag{3.15}$$

is the frequency of oscillation about the broken minimum, $\phi = \phi_c$ (see the Appendix for details) (Note: we have used a system of units where $\hbar = c = k = 1$). This is an explicit demonstration that to lowest order in $\hbar$, the thermal and quantum fluctuations of the system are treated on the same footing in the Fokker–Planck equation [10] (see the Appendix). We point out here that L itself is expected to be a function of the parameters of the theory, including temperature. However, an unfortunate consequence of neglecting the $(\nabla\phi)^2$ term from the action was the loss of all information of spatial scale. We hope that in a more sophisticated version of this model in which the effects of the spatial derivatives are properly taken into account, the value of L and the typical size of a primordial domain can be calculated explicitly.

The time independent solution (ii) and (iii) of Table 1. are dynamically derived counterparts of the flat point of the Gaussian averaged potential as seen in Fig. 1. The exact position of the stationary point is slightly shifted due to quantum/thermal fluctuations, but the effects will be clearly seen in the numerical solutions of the equations of motion (see Section 4). Finally, the solution (iv) of Table 1. is seen to assign negative values to quantities that should be positive semi– definite. This is a symptom of the fact that the mean value assignments of (3.11) are an approximation to the probability density. There is no restriction to having a positive width to the Gaussian distribution. A similar problem is seen in the averaged static potential (2.4) where the potential is a well defined for $\sigma < 0$. But, we reject such solutions of physical grounds and we do not plot that regime in Fig. 2.

In this section we have presented a dynamical model of the relaxation process based upon an empirically derived Langevin equation. The equations of motion, which assume a Gaussian distribution for the values of the scalar field, have steady state solutions in agreement with the simple static model of the second section. Encouraged by this correspondence between these two theories, we will push on to solve the equations of motion (3.12) in Section 5.

IV. Application to Some Toy Cosmologies

To this point, we could be criticized justifiably for promoting a picture that is supported only by conjecture. Although our arguments may be physically appealing, we have not provided quantitative evidence that would lead the reader to believe that our approach reflected the physics of the relaxation process any more accurately than the more conventional approaches in which the LAS equation provided the dynamical model.

To address this criticism, we will calculate the evolution of a real scalar field in a de Sitter space for the following cases:

(i) A free massive scalar field
(ii) A free massless scalar field
(iii) A massless scalar field under a $-\frac{\lambda}{4}\phi^4$ potential

We have chosen these examples because they have been investigated previously by other authors [7]. Of particular importance to our model is that these particular calculations of scalar field dynamics, took into account the fluctuations of the scalar field. Indeed, the calculation of the fluctuations of the scalar field was the focus of these articles. Therefore, we are provided with an opportunity to compare the predictions of our stochastic model with independent models and calculational schemes. As we go through the examples listed above, we shall state the results of the earlier calculations and make comparisons where appropriate. A troglodytic summary of the preceding calculations would say that these prior computations were based upon the solution of the the time development of the de Sitter modes, $\psi_k(t)$. The development of the scalar field is subsequently determined from the motion of the quantum modes. For details, the interested reader should look to the the original papers. The techniques used by these authors were based upon conventional field theory techniques, with the appropriate modification for the de Sitter geometry. From this point forward we shall refer to these earlier investigations collectively as the *conventional calculations*. We mention here that the study of fluctuations in the early universe is the subject of a growing body of literature [7,13]. We have chosen to apply our method to only a small subset of the earlier investigations.

We found that for the case of a free scalar field with $(m^2 \geq 0)$, where there exists a well defined field theory without tachyonic states, the results of the conventional calculations agreed closely with the results of our ***stochastic calculations***. To be honest, there exist differences which we attribute to an error implicit in the conventional calculations. We believe the conventional calculations contain the implicit assumption that thermal equilibrium is maintained throughout the entire history of the scalar field. Indeed, our equations of motion can be made identical to the corresponding equations of motion of the conventional models if we impose a constant value for the canonical momentum distribution (note that this value must be consistent with thermal equilibrium). Perhaps the clearest contrast between our model and those that have gone before is that — the momentum distribution of the scalar field is a dynamical variable in the stochastic model while it is implicitly assumed constant in the conventional calculations. For the case of a non–pathological scalar potential, the long time limit of the momentum fluctuations will come into agreement with the implicit assumption of the earlier calculations, and the predicted physics will be essentially the same. For this reason, there exists a high degree of unanimity between the two approaches for the case the free non–interacting scalar field.

In contrast, for cases where the field theory is plagued by tachyons (eg. a $-\lambda\phi^4$ potential), the correspondence between the conventional field theory methods and our stochastic picture becomes implausible. The resulting equations of motion are dramatically different and approach each other with only the most preposterous of assumptions. This result is physically reasonable if one considers the evolution of the momentum distribution of the scalar field. For the stable potential, the distribution will eventually stabilize and the two models will come into agreement. On the other hand, if the potential is unstable, the momentum distribution will not stabilize, or if you wish, the system will never come into thermal equilibrium. We believe that for the case of the upside–down scalar potential, the equations of motion developed earlier are wrong.

In the case of the stable theory of the free massive scalar field in de Sitter space, the characteristic temperature of the ensemble (in the infinite time limit) is found to have an upper bound. The value of the upper bound is found to be

$T_H = H/2\pi$, where T_H is the temperature that Gibbons and Hawking [14] have used to characterize the de Sitter spacetime. This result is remarkable since the methods used here are completely different from those of Gibbons and Hawking.

In the face of these comparisons and agreements with previous results, we hope that the reader will be convinced that our stochastic picture is a good representation of the dynamics of the scalar field during its evolution towards its final stable state.

4.1 The Massive Free Scalar Field

If we apply the analysis of Section 3 to a homogeneous subsection on the system the the corresponding equation of motion for the massive non–interacting scalar field is

$$\left[\frac{d^2}{dt^2} + m^2\right]\phi = -\beta\frac{d\phi}{dt} + \eta(t) \ . \tag{4.1}$$

The conversion to dimensionless variables is done according to

$$\begin{aligned} \underline{t} &\equiv \beta t \ , \\ \underline{\alpha} &\equiv \frac{m^2}{\beta^2} \ , \\ \underline{\phi} &\equiv \frac{\phi}{f} \ , \end{aligned} \tag{4.2}$$

where f is a constant of dimension one; its size will be determined by the theory. The dimensionless Langevin equation for the scalar field is then

$$\left[\frac{d^2}{d\underline{t}^2} + \frac{d}{d\underline{t}} + \underline{\alpha}\right]\underline{\phi} = \underline{\eta}(\underline{t}) \ , \tag{4.3}$$

where we assume the external force term added to simulate the effect of quantum and thermal fluctuations, $\underline{\eta}(\underline{t})$, has the usual properties associated with ***white noise*** (see (3.8) and (3.9)). Proceeding in the same manner as in Section 3, (4.3) is converted to a Fokker–Planck equation, which is then converted to the following set of ordinary differential equations:

$$\frac{d}{d\underline{t}}\langle\underline{\phi}\rangle = \langle\underline{\pi}\rangle \ ,$$

$$\frac{d}{d\underline{t}}\langle\underline{\pi}\rangle = -\langle\underline{\pi}\rangle - \underline{\alpha}\langle\underline{\phi}\rangle \ ,$$

$$\frac{d}{d\underline{t}}\langle\underline{\phi}^2\rangle = 2\langle\underline{\pi}\underline{\phi}\rangle \ ,$$

$$\frac{d}{d\underline{t}}\langle\underline{\pi}\underline{\phi}\rangle = \langle\underline{\pi}^2\rangle - \langle\underline{\pi}\underline{\phi}\rangle - \underline{\alpha}\langle\underline{\phi}^2\rangle \ ,$$

$$\frac{d}{d\underline{t}}\langle\underline{\pi}^2\rangle = -2\langle\underline{\pi}^2\rangle - 2\underline{\alpha}\langle\underline{\pi}\underline{\phi}\rangle + 2\underline{\epsilon} \ . \tag{4.4}$$

As in Section 3, $\langle\cdots\rangle$ represents an ensemble average over the set of homogeneous sub–systems. This set of ordinary differential equations is linear and thus can be solved by analytic means. In particular, the motion of $\langle\underline{\phi}^2\rangle$ is given by

$$\langle\underline{\phi}^2(\underline{t})\rangle = \frac{\underline{\epsilon}}{\underline{\alpha}} + \underline{A}e^{-\underline{t}} + \underline{B}e^{-(1+\underline{\Delta})\underline{t}} + \underline{C}e^{-(1-\underline{\Delta})\underline{t}} \ , \tag{4.5}$$

where

$$\underline{\Delta} = \sqrt{1-4\underline{\alpha}} = \sqrt{1-4\frac{m^2}{\beta^2}} \ .$$

We see that the system is driven toward a final time independent state which is characterized by

$$\lim_{t\to\infty}\langle\phi^2\rangle = f^2\frac{\underline{\epsilon}}{\underline{\alpha}} \ .$$

The analysis of a free scalar field in de Sitter space has been carried out before by Bunch and Davies [8]. They found a state with all the symmetries of the de Sitter space. In this state, usually called the Bunch–Davies vacuum, the scalar field satisfies

$$\langle\phi^2\rangle_{BD} = \frac{3H^4}{8\pi^2 m^2} \ , \tag{4.6}$$

in the limit $H \gg m$. If we assume that the ensemble of homogeneous scalar field subsystems evolves towards the properties of the Bunch–Davies vacuum then we may say that

$$f^2\underline{\epsilon} = \frac{H^2}{24\pi^2} \ , \tag{4.7}$$

where we have identified the friction term β with $3H$.

It is important to note that the inflation process does not wipe out the noise term in the Langevin equation. Naively, one might expect that a noise term might be red shifted into insignificance during the associated expansion and our analysis would become essentially the same as that of Linde, Albrecht, and Steinhardt. However, for this example the stochastic nature of the differential equations has proven to be impervious to the natural frequency shift of the de Sitter space. We believe that this feature will be true in more general situations.

The conventional calculations also predict a stable vacuum for this case and in the same manner as we have done here, fixed the constant value of the scalar field to be in agreement with the result of Bunch and Davies (4.6). It should be mentioned that, because of the nature of the conventional field theory approach, this result is obtained only after a renormalization of the parameters of the theory.

At the same time, we can consider the effective one–dimensional quantum mechanical system as embodied by the homogeneous sub–section of scalar fields. As explained in Section 3 and the Appendix, we expect that in the limit $t \to \infty$, the Fokker–Planck expectation values should take on the same values as the corresponding finite temperature quantum system. Since we assume that the system is homogeneous, the field theory problem can be treated as a one–dimensional quantum oscillator in contact with a heat bath. If we follow the arguments of the Appendix with the replacements $2\gamma \to m^2$ and $(\gamma/g) \to f^2$, we find an expression for $f^2\underline{\epsilon}$ derived from finite temperature quantum field theory:

$$f^2\underline{\epsilon} = \frac{m}{18H^2L^3}\coth\Big(\frac{m}{2T}\Big) \; . \tag{4.8}$$

It seems a safe assumption that the maximal volume of the homogeneous sub–system is limited by the horizon size; i.e.

$$L^3 \le \frac{4}{3}\pi H^{-3} \; . \tag{4.9}$$

Combining this inequality with (4.7) and (4.8) leads to

$$\frac{H^2}{24\pi^2} \ge \frac{mH}{24\pi}\coth\Big(\frac{m}{2T}\Big) \; . \tag{4.10}$$

At a large value of the temperature ($T \gg m$) the inequality (4.10) takes the form:

$$T \le \frac{H}{2\pi} = T_H \; . \tag{4.11}$$

The upper bound on the system of scalar particles is recognized as the *Hawking Temperature*, T_H. Using more formal arguments, Gibbons and Hawking have shown that in a de Sitter space, an observer will detect an omnipresent black body radiation characteristic of temperature T_H. Remarkably, it seems that the stochastic picture of the evolution correctly accounts for the effective temperature of the de Sitter space.

4.2 The Massless Free Scalar Field

The dimensionless equations of motion for a massless scalar field in de Sitter space are obtained from (4.4) by setting $\underline{\alpha}$ to zero:

$$\begin{aligned}
\frac{d}{d\underline{t}}\langle\underline{\phi}\rangle &= \langle\underline{\pi}\rangle \ , \\
\frac{d}{d\underline{t}}\langle\underline{\pi}\rangle &= -\langle\underline{\pi}\rangle \ , \\
\frac{d}{d\underline{t}}\langle\underline{\phi}^2\rangle &= 2\langle\underline{\pi}\underline{\phi}\rangle \ , \\
\frac{d}{d\underline{t}}\langle\underline{\pi}\underline{\phi}\rangle &= \langle\underline{\pi}^2\rangle - \langle\underline{\pi}\underline{\phi}\rangle \ , \\
\frac{d}{d\underline{t}}\langle\underline{\pi}^2\rangle &= -2\langle\underline{\pi}^2\rangle + 2\underline{\epsilon} \ .
\end{aligned} \tag{4.12}$$

The motion of $\langle\phi^2\rangle$ is easily solved to give:

$$\langle\underline{\phi}^2\rangle = 2\underline{\epsilon}\ \underline{t} + \underline{D} + \underline{E}e^{-\underline{t}} + \underline{F}e^{-2\underline{t}} \ . \tag{4.13}$$

Converting to dimensionful units and taking the value of $f^2\underline{\epsilon}$ from the $m^2 > 0$ case (4.7), the evolution of the massless scalar field is given by

$$\langle\phi^2\rangle = \frac{H^3}{4\pi^2}t + D + Ee^{-3Ht} + Fe^{-6Ht} \ . \tag{4.14}$$

At the same time, the momentum fluctuations are found to evolve towards a steady state value according to

$$\langle\pi^2(t)\rangle = \frac{3H^4}{8\pi^2} + Ge^{-6Ht} \ . \tag{4.15}$$

The vacuum of the massless scalar field is found to be unstable in de Sitter space, however the system proceeds toward a thermal equilibrium state in a time of the order $\sim (6H)^{-1}$.

To see the contrast of the stochastic method with the conventional model, consider the equation of motion for the scalar field as derived by the conventional model [7]:

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt}\right)\langle\phi^2\rangle = \frac{3H^4}{4\pi^2} , \tag{4.16}$$

and the solution is,

$$\langle\phi^2\rangle = \frac{H^3}{4\pi^2}t + D' + E'e^{-3Ht} . \tag{4.17}$$

The equations of motion and the solutions for the stochastic and conventional models are very similar for this case of the massless scalar field. The linear growth term for the scalar field $(H^3/4\pi^2)t$ is identical to the coefficient found in our stochastic calculations. Indeed, if we set $\langle\underline{\pi}^2\rangle = \underline{\epsilon}$ in the stochastically based equations (4.12), the resulting equations and solution for $\langle\phi^2\rangle$ are identical to the equations and solution determined by the conventional methods (4.16). The most natural interpretation of this connection between the two approaches of the scalar dynamics it that the conventional equations of motion represent the long time limit, or if you wish the thermal equilibrium limit of the corresponding equations of motion based upon the stochastic model. We emphasize that this coincidence occurs because the inherent field theory associated with the free scalar field is free of pathologies such a tachyons.

4.3 The Scalar Field Under an Unstable Potential

Not all potentials lead to agreement between the stochastic and conventional models of scalar field dynamics. The general rule is that if the scalar potential leads to a momentum distribution which eventually stabilizes at a value characteristic of thermal equilibrium, then the two approaches will share many of the same features. On the other hand, if the scalar field is unstable, such that the momentum fluctuations do not have a well behaved asymptotic behavior, then the conventional approach, which implicitly assumes thermal equilibrium, will not resemble the results of the stochastic approach. This is not surprising when one realizes that such potentials also have field theories that are fundamentally sick, where the spectrum has negative mass states. Conventional quantum calculations in such situations are suspect from the beginning.

The Coleman–Weinberg scalar potential [9] near small values of the scalar field has the form of and upside–down quartic scalar potential.

$$V_{CW} \sim -\frac{\lambda}{4}\phi^4 \quad \text{for} \quad |\,\phi\,| \approx 0\ . \tag{4.18}$$

The associated Langevin equation for such potential is,

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt}\right)\phi = -\lambda\phi^3 + \eta(t)\ . \tag{4.19}$$

If we define a dimensionless coupling by,

$$\underline{\alpha} \equiv \frac{\lambda f^2}{9H^2}\ , \tag{4.20}$$

where f is again the scale of the scalar field as in (4.2). Following the procedures of Section 3 along with the Hartree–Fock approximation for the scalar field distribution, the dimensionless Fokker–Planck equations of motion for the ensemble expectation values are:

$$\frac{d\langle\underline{\phi}\rangle}{d\underline{t}} = \langle\underline{\pi}\rangle\ ,$$

$$\frac{d\langle\underline{\pi}\rangle}{d\underline{t}} = -\langle\underline{\pi}\rangle + \underline{\alpha}\langle\underline{\phi}\rangle\Big(3\langle\underline{\phi}^2\rangle - 2\langle\underline{\phi}\rangle^2\Big)\ ,$$

$$\frac{d\langle\underline{\phi}^2\rangle}{d\underline{t}} = 2\langle\underline{\pi}\underline{\phi}\rangle\ ,$$

$$\frac{d\langle\underline{\pi}^2\rangle}{d\underline{t}} = -2\langle\underline{\pi}^2\rangle + 2\underline{\epsilon} + 2\underline{\alpha}\Big(3\langle\underline{\pi\phi}\rangle\langle\underline{\phi}^2\rangle - 2\langle\underline{\pi}\rangle\langle\underline{\phi}\rangle^3\Big) ,$$

$$\frac{d\langle\underline{\pi\phi}\rangle}{d\underline{t}} = \langle\underline{\pi}^2\rangle - \langle\underline{\pi\phi}\rangle + \underline{\alpha}\Big(3\langle\underline{\phi}^2\rangle^2 - 2\langle\underline{\phi}\rangle^4\Big) . \tag{4.21}$$

In general, these equations must be solved by numerical methods. However, we have assumed that $|\langle\underline{\phi}\rangle| \ll 1$ to obtain the approximation to the Coleman–Weinberg potential given by (4.18). In this approximation the Fokker–Planck equations (4.21) become:

$$\frac{d}{d\underline{t}}\Big(\frac{d}{d\underline{t}} - 1\Big)\langle\underline{\phi}\rangle \simeq 0 ,$$

$$\Big(\frac{d}{d\underline{t}} + 1\Big)\langle\underline{\pi}\rangle \simeq 0 ,$$

$$\frac{d\langle\underline{\phi}^2\rangle}{d\underline{t}} = 2\langle\underline{\pi\phi}\rangle ,$$

$$\frac{d\langle\underline{\pi}^2\rangle}{d\underline{t}} \simeq -2\langle\underline{\pi}^2\rangle + 2\underline{\epsilon} + 6\underline{\alpha}\langle\underline{\pi\phi}\rangle\langle\underline{\phi}^2\rangle ,$$

$$\frac{d\langle\underline{\pi\phi}\rangle}{d\underline{t}} \simeq \langle\underline{\pi}^2\rangle - \langle\underline{\pi\phi}\rangle + 3\underline{\alpha}\langle\underline{\phi}^2\rangle^2 \tag{4.22}$$

If we assume that the momentum distribution is at an equilibrium value i.e. $d\langle\underline{\pi}^2\rangle/dt = 0$, then the equation of motion for the scalar field is

$$\Big(\frac{d^2}{dt^2} + 3H\frac{d}{dt}\Big)\langle\phi^2\rangle \simeq \frac{3H^4}{4\pi^2} + 6\lambda\langle\phi^2\rangle^2 + \frac{\lambda}{2H}\frac{d}{dt}\langle\phi^2\rangle^2 , \tag{4.23}$$

where we have used the strength of the stochastic noise term determined in the free scalar field cases (4.7).

In contrast, Vilenkin found the equation of motion to be:

$$\Big(\frac{d^2}{dt^2} + 3H\frac{d}{dt}\Big)\langle\phi^2\rangle \simeq \frac{3H^4}{4\pi^2} + 6\lambda\langle\phi^2\rangle^2 \tag{4.24}$$

In order to make these two equations of motion consistent, we would have to set $\langle\phi^2\rangle$ to a purely imaginary constant. In fact, we cannot find any physically consistent way in which we can convert our stochastically derived equations of motion to be consistent with the equations of motion of the conventional method. This incompatibility between the two methods is to be expected because of the pathological nature of the potential.

V. Solving Numerically the Equations of Motion

In order to have numerical results in a form that can be understood on an intuitive level, we shall adopt the conventions of the static model of Section 2. Therefore, we will change the variables in the Fokker–Planck equation (3.12) to:

$$\underline{\sigma} \equiv \langle \underline{\phi}^2 \rangle - \langle \underline{\phi} \rangle^2 , \tag{5.1}$$

$$\underline{\sigma}_\pi \equiv \frac{1}{\underline{\epsilon}}(\langle \underline{\pi}^2 \rangle - \langle \underline{\pi} \rangle^2) . \tag{5.2}$$

$\underline{\sigma}$ is the same object introduced in the second section as a parameter in the static potential $\underline{U}$, however in this section it is a dynamical variable. In terms of $\langle \underline{\phi} \rangle$ and these alternate variables $\underline{\sigma}$ and $\underline{\sigma}_\pi$, the Fokker–Planck equations (3.13) lead to:

$$\begin{aligned}
\frac{d^2}{d\underline{t}^2}\langle \underline{\phi} \rangle + \frac{d}{d\underline{t}}\langle \underline{\phi} \rangle &= \underline{\alpha}\langle \underline{\phi} \rangle(1 - 3\underline{\sigma} - \langle \underline{\phi} \rangle^2) , \\
\frac{d^2}{d\underline{t}^2}\underline{\sigma} + \frac{d}{d\underline{t}}\underline{\sigma} &= 2\underline{\alpha}\, \underline{\sigma}(1 - 3\underline{\sigma} - 3\langle \underline{\phi} \rangle^2) , \\
\frac{d}{d\underline{t}}\underline{\sigma}_\pi &= 2(1 - \underline{\sigma}_\pi) + \frac{\underline{\alpha}}{\underline{\epsilon}}(1 - 3\underline{\sigma} - 3\langle \underline{\phi} \rangle^2) .
\end{aligned} \tag{5.3}$$

There are seven real parameters which determine the character of the solution of the Fokker–Planck equations — five initial conditions consisting of the values of $[\langle \underline{\phi} \rangle,\ d\langle \underline{\phi} \rangle/dt,\ \underline{\sigma},\ d\underline{\sigma}/dt,\ \underline{\sigma}_\pi]$ at $\underline{t} = 0$; and the parameters $\underline{\epsilon}$ and $\underline{\alpha}$.

During our numerical investigations, we were able to investigate only a small portion of the seven dimensional parameter space of the theory. In the examples that we will show here, we set the initial conditions of the dynamical variables to be:

$$\langle \underline{\phi} \rangle_0 = 10^{-3}, \qquad \frac{d}{d\underline{t}}\langle \underline{\phi} \rangle_0 = \langle \underline{\pi} \rangle_0 = 0 ,$$

(there is nothing special about these values other than they are small and they bias the system toward the $\underline{\phi} = 1$ vacuum)

$$\underline{\sigma}_0 = \left(\frac{d}{d\underline{t}}\underline{\sigma}\right)_0 = 0 ,$$

(the system is assumed initially homogeneous)

$$\langle \underline{\sigma}_\pi \rangle = 0 ,$$

(numerical results are found to be insensitive to this parameter — the physics of the system is dependent upon the temperature of the heat bath (3.14) rather than the initial temperature of the system). In addition, we have fixed the dimensionless dimensionless constant $\underline{\alpha}$ to be 1. This arbitrary choice is weakly justified by our conjecture, and observation that a different value of $\underline{\alpha}$ will not unearth any dramatic new behavior of the variables. That is, by varying $\underline{\epsilon}$ alone, it should be possible to see all the important qualitative features of the solutions of the Fokker–Planck equations. Of course, it would be nice to perform a more extensive exploration of the parameter space, but we will postpone that to a future investigation.

We will show in this article the results of just two points in the seven dimensional parameter space. However, in the course of our numerical investigation, we looked at many points in the parameter space and we found that all the observed results could be assigned into one of two categories. The two points in the parameter space were chosen to demonstrate each category of relaxation. We will see that the predictions of the earlier sections about the qualitative nature the relaxation process will be confirmed.

In Figure 5 we show $\langle \underline{\phi} \rangle$ versus $\underline{t}$ for three cases. The curve labeled LAS is the solution for the equations of motion where the fluctuations have been set to zero. This is the motion that would be predicted by the simple fluctuation free rolldown picture. The curve labeled **A** is a solution of the Fokker–Planck equations (5.3) with $\underline{\epsilon} = 10^{-5}$, the curve labeled **B** is a solution for $\underline{\epsilon} = 10^{-4}$. The initial conditions and $\underline{\alpha}$ are as specified earlier. The **A** curve is close to the LAS curve and is analogous to the curve marker **A** in Fig. 4. A conspicuous period of delayed development is not observed for this value of $\underline{\epsilon}$. The curve labeled **B** is dramatically different. The scalar field has a small initial increase which is followed by a relatively slow growth to the final value of 1, the vacuum state. This delay in development is the dynamical analogue of the delayed behavior of the static model as shown in the **B** path of Fig. 4. We have found that by increasing the value of $\underline{\epsilon}$ this period of delay could be made indefinitely long. Indeed, in many cases the system would become stuck in this delay state, not relaxing to the vacuum state before the computation was complete. We interpret this as meaning that the friction term in the equation of motion is effectively large enough to make the solution of the system near the delay point overdamped, and the delay time becomes long (of order $\underline{\alpha}/\underline{\epsilon}$).

In Fig. 6 we plot $\underline{\sigma}$ versus $\underline{t}$. It is interesting to note that during the point of delay of curve **B**, the value of $\underline{\sigma}$ is close to 1/3 as predicted by the static model of Section 2. It appears that the friction term associated with curve **A** is not large enough to hold the system at the flat part of the static potential (2.4).

In Fig. 7 we plot $\underline{\sigma}_\pi$ versus $\underline{t}$. The scales of the two curves are seen to be dramatically different. It should be mentioned that the equilibrium value of $\underline{\sigma}_\pi$ is equal to 1 in both cases in agreement with the analysis of Section 3, but it is too small to be seen on the figure.

VI. Discussion and Conclusion

In this article we have presented a picture of the inflation process in which spatial fluctuation in the scalar field play an important role in the dynamics of the process. A simple back of the envelope calculation was presented in Section 2. The results of that section lead to the conclusion that it is reasonable that at the dynamics of the inflation process will have (at points where the scalar potential is not convex) a tendency to enhance the spatial fluctuation of the scalar field controlling inflation. Further, during a period of large fluctuation, the development of the scalar field will be slowed, thereby trapping the universe in a state of non-zero cosmological constant for a period that could be exponentially long.

If the scenario of a long period of significant spatial fluctuations is to be taken seriously, then the concept of a well defined, spatially homogeneous bubble slowly rolling down an effective potential must be given up. Indeed we suggest that the fluctuations will be so large that the concept of one or many bubbles is an inappropriate view of the relaxation process. These types of criticisms are not new. Mazenko, Unruh, and Wald [2] argued that the universe will not follow the roll down picture of the New Inflationary Scenario [1], rather, the system "quickly" fractures into primordial domains thereby prematurely terminating the inflation process. Our analysis leads to a picture that is similar to that of MUW; we reject the concept of bubbles as a description of the symmetry breaking process (relaxation). Relaxation, is best viewed as the transition from an initial state (that has energy that is above the minimum value of the ground state) to a state where the system has settled into primordial domains of degenerate vacua. Our view differs from that of MUW because we believe that the transition from the initial state to the state dominated by microscopic domains may require a long period of time in which the average of the stress energy tensor is nonzero. We believe that it may be possible that sufficient inflation occurs during this period of spatial fluctuation of the scalar field to encompass the successes of the original inflationary scenarios. We believe that the criticisms of MUW of the NIS are valid, but the attractive idea of inflationary scenario may survive the scrutiny of an analytical study of the dynamics of the symmetry breaking process with all the good parts still intact; i.e.

the solution of the horizon–, monopole–, and flatness–problems of cosmology.

In Section 2 we presented a static model of the effect of spatial fluctuations upon the relaxation dynamics. In our model, it is the distribution scalar field values in the the system ($W(\phi)$) that plays the role of the fundamental dynamical variable. For the sake of simplicity, we assumed that the distribution to be of a Gaussian nature. We then calculated an average static energy of a system whose spatial fluctuations are assumed to have negligible effect upon the energy. We had therefore excluded the presence of domain walls from the system. The Gaussian distribution and therefore the state of the system is characterized by two parameters, $\langle\underline{\phi}\rangle$ and $\underline{\sigma}$. The shape of the potential in this two dimensional space was plotted in Fig. 4. From this, many of the qualitative features the relaxation process are then deduced easily.

In Section 3 we presented a dynamical model of the relaxation process. The spatial sub–section of interest is assumed to be so small that spatial variations of the scalar field are assumed to be of no importance to the dynamical process, therefore the classical equation of motion of the scalar field took on a form similar to the Linde–Albrecht– Steinhardt equation of motion for the scalar field. However, even though we assumed that the inflation process is smooth on a small enough scale, there is no guarantee that on a larger scale, the process is totally homogeneous. Indeed, in Section 2, we gave arguments that it is unreasonable to assume that the entire universe remains completely homogeneous during the entire relaxation. Therefore in order to simulate the effects of quantum and thermal fluctuations, we added to the LAS equations of motion a white noise term, $\eta(t)$. The resulting stochastic LAS equation is immediately recognized as a Langevin equation for the ensemble of scalar field sub– systems that make up the universe. Using standard methods, the stochastic LAS equation for ϕ is converted into a partial differential equation for $W(\pi, \phi, t)$; the probability distribution in the canonical phase space of the scalar field. Such an equation is commonly referred to as a Fokker–Planck or Kramers equation. With the additional assumption that the distribution has a Gaussian shape in the ϕ–direction in the phase space (a similar assumption is made in the Static model of Section 2) the partial equation of motion are converted to a set of five coupled first– order non–linear ordinary differential equations in the ensemble expectation values of $\langle\phi\rangle, \langle\pi\rangle, \langle\phi^2\rangle, \langle\pi^2\rangle$, and $\langle\pi\phi\rangle$. The equations of motion are then examined for their stationary points and the results (shown in detail in Table 1) agree with the static model in that they show that a slowing in the development of the scalar field occurs at a point where the fluctuations of the scalar field are enhanced. Finally, the long time limit of the Fokker–Planck solutions are compared to the expectation values of an analogous finite temperature quantum system. The comparison between the two approaches leads to an explicit relationship between the strength of the white noise term in the Langevin equation for the scalar field and the temperature of a corresponding system of quantum scalar fields.

In order to test the validity of the stochastic model of dynamics developed in Section 3, we calculated the evolution of the scalar field for some *Toy Cosmologies* in Section 4. The evolution of the scalar field for these cases have been previously studied by other authors. These earlier studies used more conventional calculational techniques based upon field theory. We found that for cases where the corresponding field theory is well defined (i.e. the spectrum does not include particles of negative mass), the results of our stochastic model and the earlier work are in close agreement. For cases where the potential does not give rise to a well defined field theory, the results of the earlier conventional calculation do not resemble the results of our stochastic method. The two calculational techniques will normally come into agreement only in the infinite time limit of the Langevin equation, where the field configurations take on the form characteristic of thermodynamic equilibrium. At this point, the solutions of the stochastic differential equations should be identical to the results of finite temperature quantum field theory (which implicitly assumes that the physical system is in thermodynamic equilibrium). For ill behaved potentials, (e.g. $-\lambda\phi^4$) the system cannot usually come into equilibrium and the two methods cannot be expected to come into agreement.

In addition to the comparisons with the results of the scalar field evolution, a particularly interesting result is observed for a free massive scalar field. The characteristic temperature of an ensemble of homogeneous sub–systems is bounded above by $T = H/2\pi$, a temperature that Gibbons and Hawking have associated with the de Sitter space [14]. The agreement of the results is remarkable since our techniques are so different than the original approach to calculating the temperature. With this succession of agreements with earlier calculations, we are confident that our stochastic method accurately reflects the physics of the relaxation process.

We next presented the results of the numerical solutions of the coupled differential equations developed in Section 3. We focused our attention upon two contrasting cases. One case resembles the motion predicted by the LAS equation (except for a short period in which the fluctuations are large) while the other has a long delay period of large fluctuations in the value of the scalar field. We have investigated a reasonably large portion of the parameter space of the model and always found the solutions to fall into one of these two classes. The qualitative nature of the solutions are in agreement with the conclusions derived from the Static Model of the Second Section. It should be emphasized that in either case, there exists a period where the fluctuations of the scalar field are so large that the concept of a "bubble" becomes an inappropriate description of the process of relaxation.

The stochastic models that we have suggested are not complete. As we see it, the two most obvious short comings are: (1) The energy associated with spatial derivatives have been neglected. The result is that the theory has no scale other that H^{-1}, the inverse of Hubble's constant. We therefore cannot determine the size of the homogeneous domains created in the final stages of the relaxation process. (2) We have assumed that the shape of the distribution of scalar field values is Gaussian. This precludes a double peaked distribution centered around the minima of the scalar potential which is the distribution associated with the entire universe in the final stages of relaxation where the universe has divided itself into primordial domains. It is possible that if we had allowed for such a distribution on all scales, the resulting dynamical picture could have been different. Both of these points will be addressed in future investigations.

Despite the shortcomings that we have noted, we believe that the stochastic models presented in this article are simple and intuitive. From comparison with earlier calculations, we believe that the evolution of the scalar field is accurately described in a number of important cases. Certainly, fluctuations in the value of the scalar field will have a dramatic effect on the dynamical picture. We hope that this article represents a useful approach to the understanding of the evolution toward equilibrium.

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Appendix

In the Bjorken and Drell system of units, the Langevin equation for the scalar field is

$$\frac{d^2}{dt^2}\phi = -\beta\frac{d}{dt}\phi + \gamma\phi - g\phi^3 + \eta(t)\ , \tag{A.1}$$

Indeed, this is the equation of motion for a typical homogeneous sub–section of the heterogeneous system (the observed universe). The purpose of the Langevin equation is to simulate the effect of quantum and thermal fluctuations upon the field values of the system. If the simulation were true, then the result of the steady state solution of the Langevin equation should agree with the results of finite temperature quantum field theory. In the steady state limit of the Langevin equation, the energy loss due to the friction term is exactly balanced by the energy gain by the external noise term. Our strategy will be to find a finite temperature quantum system with the same value of energy and expectation values of field operators. From this point of view, the term $\beta d\phi/dt$ is not regarded as symptom of spatial curvature, but simply a friction term in a flat space.

The Hamiltonian density of a typical homogeneous sub–system is then

$$\tilde{H} = \frac{1}{2}\Big(\frac{\partial\phi}{\partial t}\Big)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\ . \tag{A.2}$$

Since the sub–system is, by fiat, homogeneous, the derivative term can be neglected within this approximation. The resulting spatially independent Hamiltonian density is (A.2) with the spatial derivative term removed. The proper Hamiltonian of the sub–system H, is the spatial integral of the Hamiltonian density $\tilde{H}$.

$$H(t) \equiv \int \tilde{H}(t)\ d^3x\ . \tag{A.3}$$

We define one dimensional quantum operators for this simplified system as the following:

$$\Phi(t) \equiv \frac{1}{\sqrt{L^3}}\int d^3x\Big(\phi - \sqrt{\frac{\gamma}{g}}\Big)\ , \tag{A.4}$$

$$\Pi(t) \equiv \frac{d}{dt}\Phi(t) \equiv \frac{1}{\sqrt{L^3}}\int d^3x\ \pi(t)\ , \tag{A.5}$$

where L^3 is the volume of the homogeneous sub–system. The commutation relations for Φ and Π follow from the fundamental equal time commutation relations for ϕ and π,

$$\left[\Phi(t), \Pi(t)\right] = \frac{1}{L^3} \int\int d^3x d^3y [\phi(\vec{x}, t), \pi(\vec{y}, t)] = i \ . \tag{A.6}$$

The upshot is that by assuming that the sub–system of the Langevin equation is homogeneous, the sub–system is effectively a one–dimensional quantum mechanical system with a Hamiltonian of the form

$$H = \frac{1}{2}\Pi^2 + \gamma\Phi^2 - \frac{\gamma^2}{4g}L^3 + \frac{\sqrt{\gamma g}}{\sqrt{L^3}}\Phi^3 + \frac{g}{4L^3}\Phi^4 \ . \tag{A.7}$$

If g is assumed to be small and if we are willing to ignore the dangers associated with the renormalization of a scalar theory, then the Φ^3 and Φ^4 terms may be dropped from the effective one–dimensional Hamiltonian (A.7). The result is a theory of a one–dimensional harmonic oscillator with frequency $\omega = \sqrt{2\gamma}$. At finite temperature T, the expectation value of any operator X is,

$$\langle X \rangle_T \equiv \frac{Tr\left[e^{-H/T} X\right]}{Tr\left[e^{-H/T}\right]} \ . \tag{A.8}$$

If the stochastic simulation of this quantum system (as described in Section 3) is correct, then the $t \to \infty$ limit of the Fokker–Planck (FP) expectation value of any quantity must take on the same value as obtained by conventional equilibrium statistical mechanical methods; that is,

$$\langle X \rangle_T = \lim_{t \to \infty} \langle X \rangle_{FP} \ . \tag{A.9}$$

The application of this equality to $\langle \underline{\pi}^2 \rangle$ (see (3.14)) yields,

$$\underline{\epsilon} = \frac{g}{\sqrt{2\gamma}\beta^2 L^3} \coth\left(\frac{\sqrt{2\gamma}}{2T}\right) \ . \tag{A.10}$$

If we reinstate the hidden factors of $\hbar$, c, and k, then the dimensionless coefficient of the Gaussian noise term is given by,

$$\underline{\epsilon} = \frac{\hbar g}{\sqrt{2\gamma}\beta^2 L^3} \coth\left(\frac{\sqrt{2\gamma}}{2}\frac{\hbar c}{kT}\right) \ . \tag{A.11}$$

It is in the sense made explicit by (A.11) that both thermal and quantum fluctuations are accounted for by the inclusion of a Gaussian noise term ($\underline{\eta}(t)$) into the equations of motion for the scalar field.

REFERENCES

1. A. D. Linde, Phys. Lett. 108B (1982) 389;
   A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48 (1982) 1220.
   Reviews of the subject are presented in:
   A. D. Linde, Rep. Prog. Phys. 47 (1984) 925;
   R. H. Brandenberger, Rev. Mod. Phys. 57 (1985) 1;
   Michael S. Turner, *The Inflationary Paradigm*, Fermilab preprint (1985).

2. G. F. Mazenko, W. G. Unruh, and R. M. Wald, Phys. Rev. D31 (1985) 273.

3. M. Evans and J. G. McCarthy, Phys. Rev. D31 (1985) 1799.

4. J. S. Langer in Fluctuations, Instabilities, and Phase Transitions (Plenum Press, New York, 1975) p. 19.

5. J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D10 (1974) 2428.

6. S. W. Hawking and I. G. Moss, Nucl. Phys. 224B (1983) 180.

7. A. Vilenkin, Nucl. Phys. 226B (1983) 504; ibid. 527;
   A. Vilenkin and L. H. Ford, Phys. Rev. D26 (1982) 1231;
   A. D. Linde, Phys. Lett. 116B (1982) 335.

8. T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. (London) A360 (1978) 117.

9. S. Coleman and E. J. Weinberg, Phys. Rev. D7 (1973) 1888.

10. A. O. Caldeira and A. J. Legget, Physica 121A (1983) 587;
    H. Decker, Phys. Rev. A16 (1977) 2116.

11. G. Parisi and N. Sourlas, Phys. Rev. Lett. 43 (1979) 744;
    G. Parisi and Yongshi Wu, Sci. Sin. 24 (1981) 483.

12. S. Chandrasekhar, Rev. Mod. Phys. 15 (1943) 1;
    H. Risken, The Fokker-Planck Equation (Springer—Verlag, New York, 1984).

13. A. A. Starobinsky, Phys. Lett. 117B (1982) 175;
Alan H. Guth and So–Young Pi, Phys. Rev. Lett. 49 (1982) 1110;
Alan H. Guth and So–Young Pi, *The Quantum Mechanics of the Scalar Field in the New Inflationary Scenario*, MIT reprint (1985);
S. W. Hawking, Phys. Lett. 115B (1982) 295;
J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D28 (1983) 679;
R. Brandenberger, R. Kahn, W. H. Press, Phys. Rev. D28 (1983) 1809;
R. Brandenberger and R. Kahn, Phys. Rev. D29 (1984) 2172.

14. G. Gibbons and S. W. Hawking, Phys. Rev. D15 (1977) 2738;
S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.

TABLE 1

$\lim\limits_{\frac{\epsilon}{\underline{\alpha}}\to 0+}$	$\langle\underline{\phi}\rangle^2$	$\langle\underline{\phi}^2\rangle$	$\underline{\sigma} \equiv \langle\underline{\phi}^2\rangle - \langle\underline{\phi}\rangle^2$
(i)	$\frac{1+\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{2} \longrightarrow 1-\frac{3}{2}\frac{\epsilon}{\underline{\alpha}}$	$\frac{2+\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{3} \longrightarrow 1-\frac{\epsilon}{\underline{\alpha}}$	$\frac{1-\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{6} \longrightarrow \frac{1}{2}\frac{\epsilon}{\underline{\alpha}}$
(ii)	$\frac{1-\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{2} \longrightarrow \frac{3}{2}\frac{\epsilon}{\underline{\alpha}}$	$\frac{2-\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{3} \longrightarrow \frac{1}{3}+\frac{\epsilon}{\underline{\alpha}}$	$\frac{1+\sqrt{1-6\frac{\epsilon}{\underline{\alpha}}}}{6} \longrightarrow \frac{1}{3}-\frac{1}{2}\frac{\epsilon}{\underline{\alpha}}$
(iii)	0	$\frac{1+\sqrt{1+12\frac{\epsilon}{\underline{\alpha}}}}{6} \longrightarrow \frac{1}{3}+\frac{\epsilon}{\underline{\alpha}}$	$\frac{1+\sqrt{1+12\frac{\epsilon}{\underline{\alpha}}}}{6} \longrightarrow \frac{1}{3}+\frac{\epsilon}{\underline{\alpha}}$
(iv)	0	$\frac{1-\sqrt{1+12\frac{\epsilon}{\underline{\alpha}}}}{6} < 0$	$\frac{1-\sqrt{1+12\frac{\epsilon}{\underline{\alpha}}}}{6} < 0$

Steady state solutions of the stochastic equations of motion.

At the same time: $\langle\underline{\pi}\rangle = \langle\underline{\pi}\underline{\phi}\rangle = 0$ and $\langle\underline{\pi}^2\rangle = \underline{\epsilon}$.

Figure Captions

1. A possible initial configuration for the universe in which $\phi = 0$ everywhere. The small square in the center depicts the homogeneous sub–systems discussed in Section 3.

2. The universe passes through a state of high fluctuation in the process of relaxation. Bubbles are not well defined.

3. In the final stage of the relaxation process, the universe has settled into primordial domains of $\phi \simeq \phi_c$. The square in the middle of the figure represents the observed universe.

4. The effective static potential $\underline{U}(\langle\underline{\phi}\rangle, \underline{\sigma})$. For the purpose of visual clarity, the potential has been cut off at a maximum value of zero. The flat sections on the corners are an artifact of this cut–off and should be ignored. Typical relaxation scenarios are depicted as paths upon this two dimensional surface.

5. The result of numerical solutions of the stochastic equations of motion are shown. A plot of $\langle\underline{\phi}\rangle$ versus $\underline{t}$ for various conditions show the two different classes of possible relaxation scenarios.

6. The result of the numerical calculation for $\underline{\sigma}$ versus $\underline{t}$. There is a period of delay for curve **B** at the value of $\underline{\sigma} \simeq \frac{1}{3}$ in agreement with the static and dynamic pictures of the relaxation process.

7. Numerical results of $\underline{\sigma}_\pi$ versus $\underline{t}$.

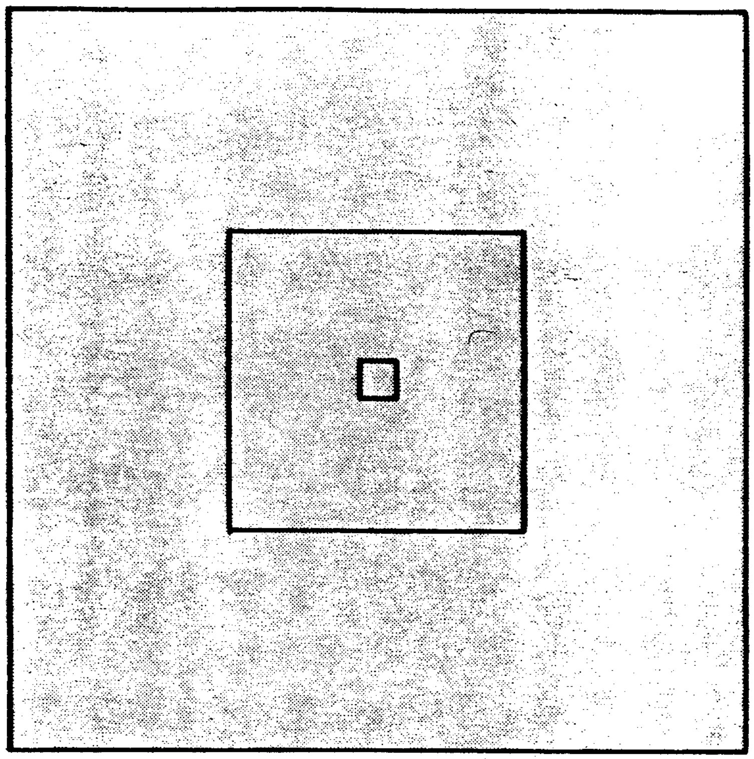

Fig. 1

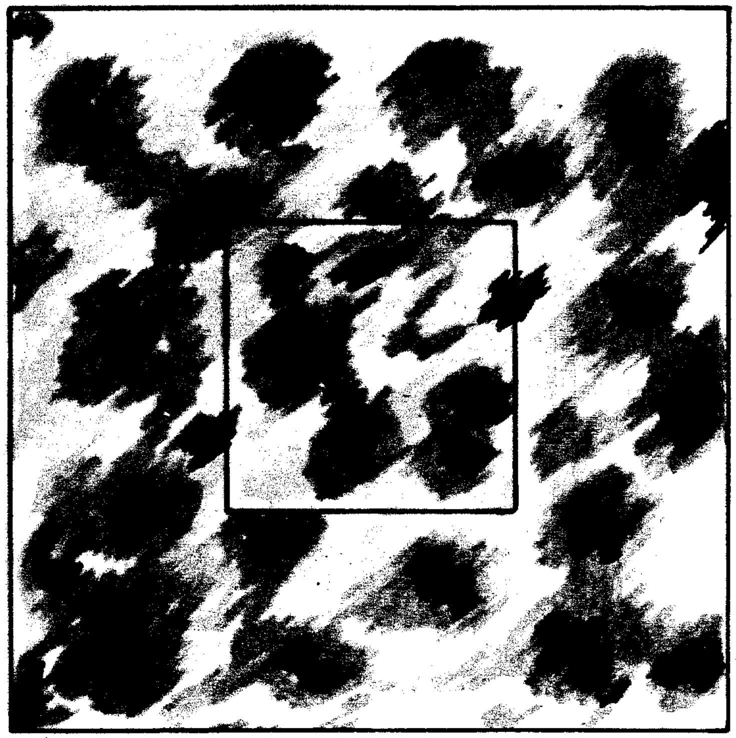

Fig.2

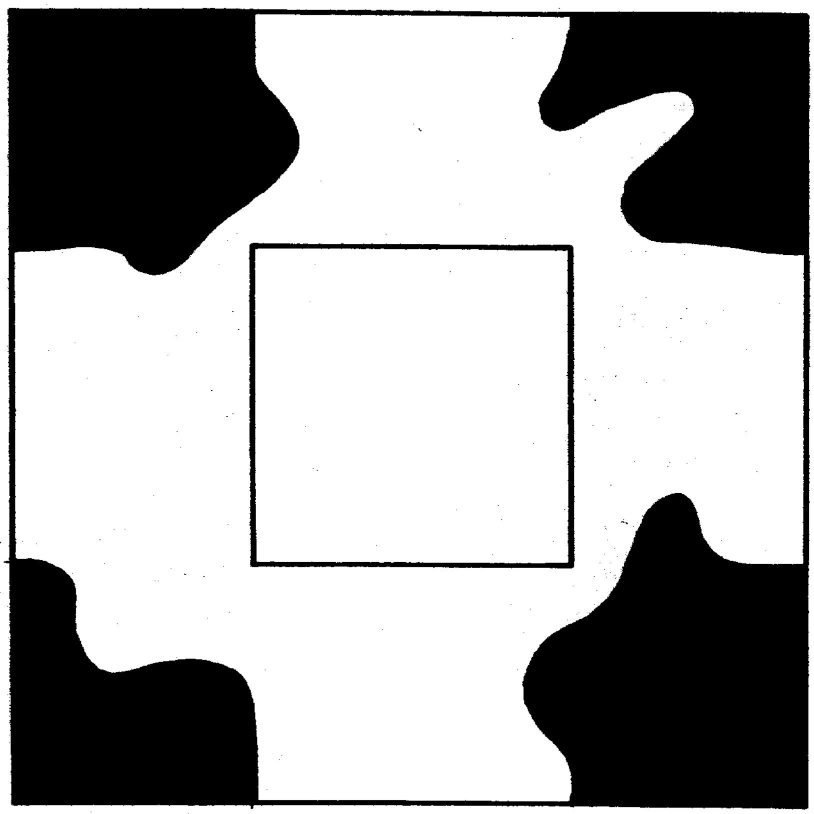

Fig.3

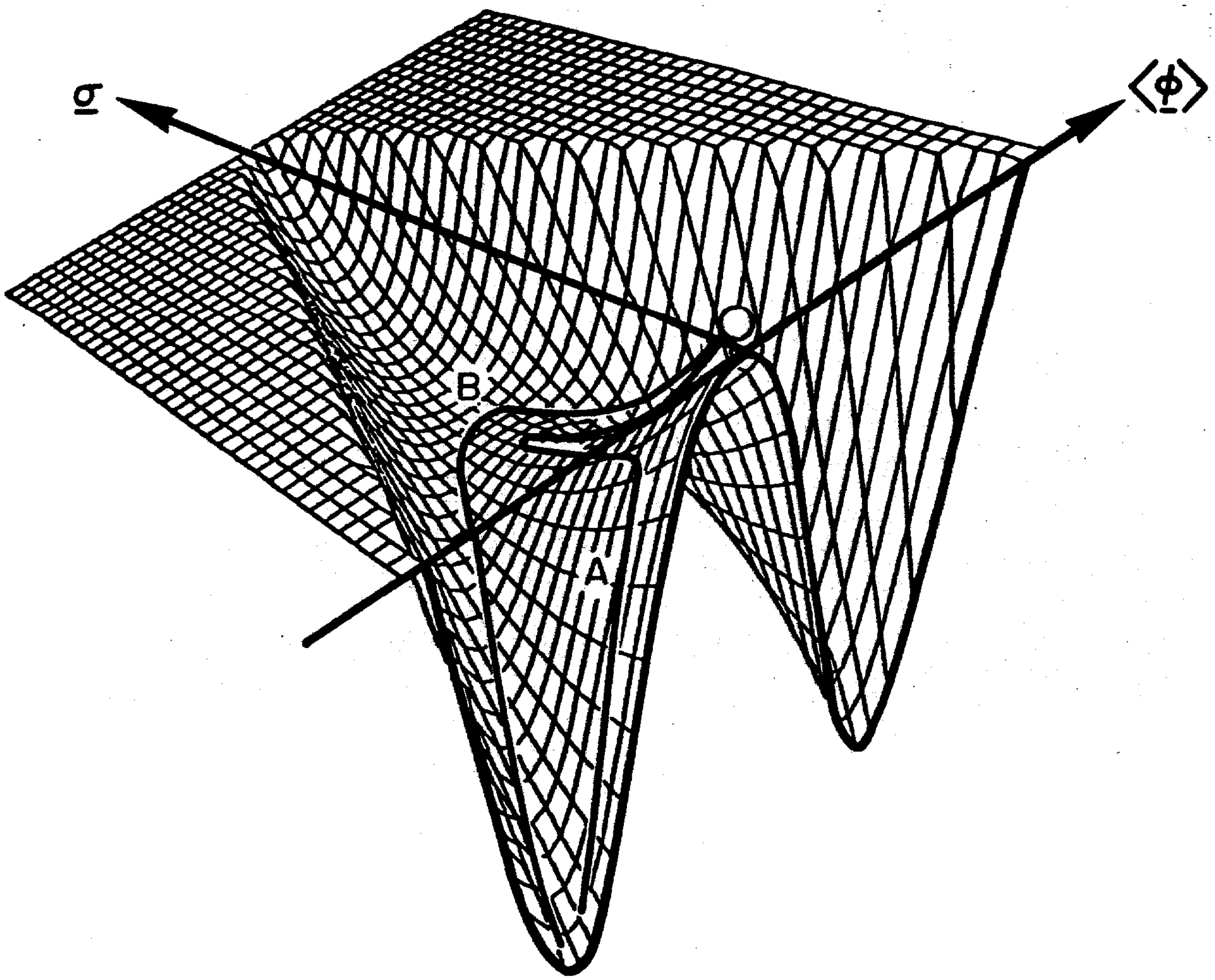

Fig. 4

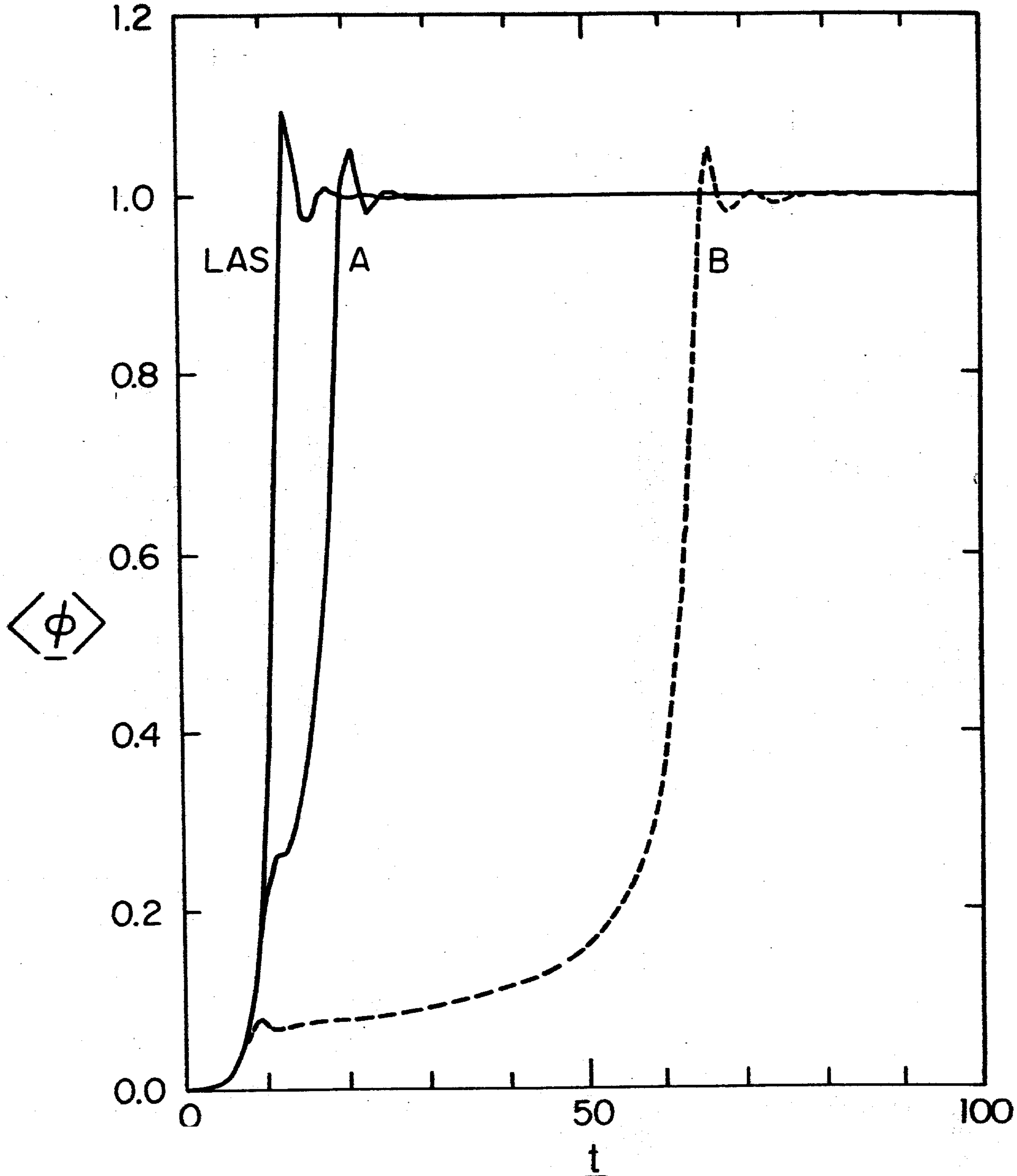
1.2
1.0
0.8
0.6
0.4
0.2
0.0
$\langle \underline{\phi} \rangle$
LAS
A
B
0
50
100
$\underline{t}$

Fig.5

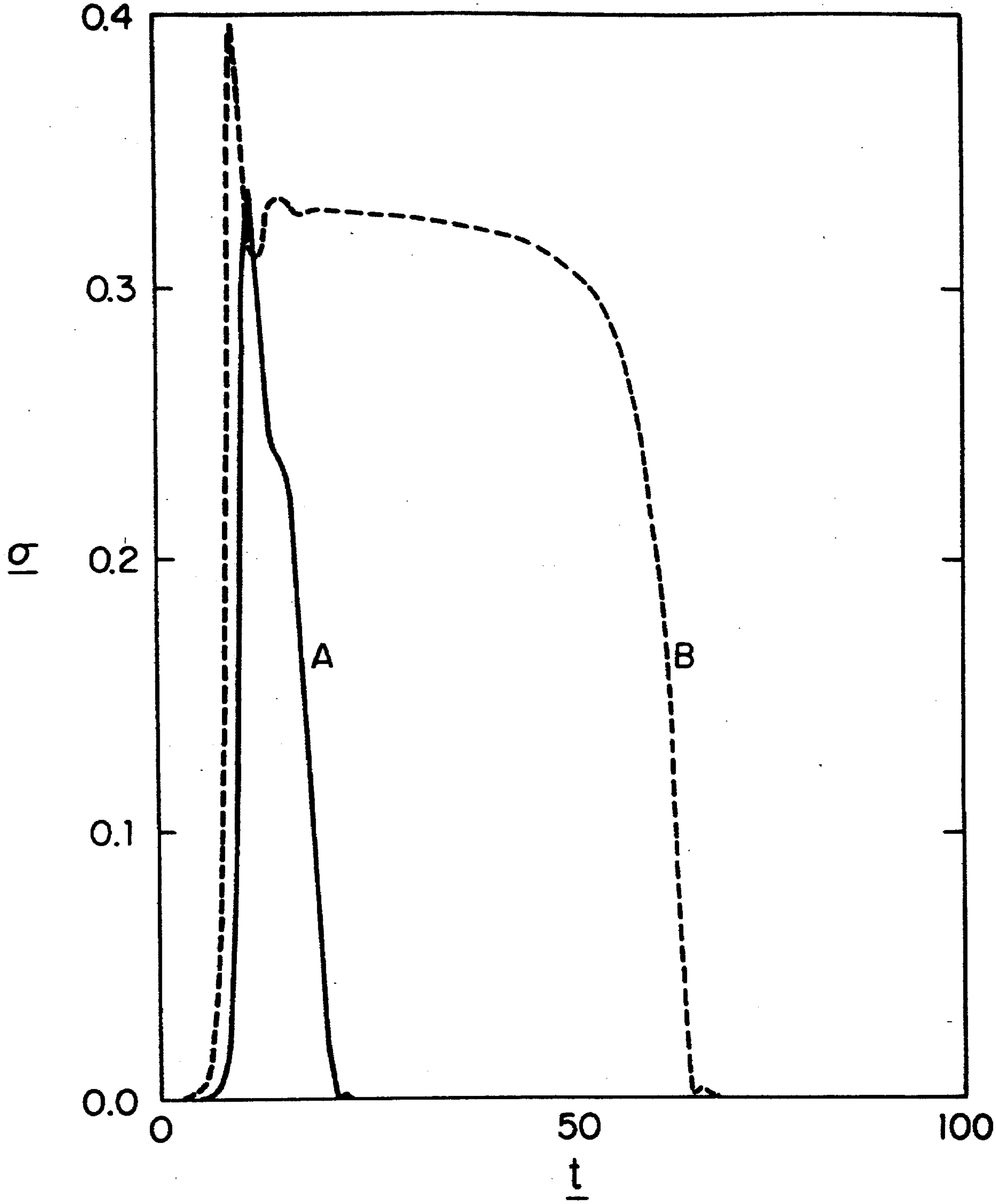

Fig.6

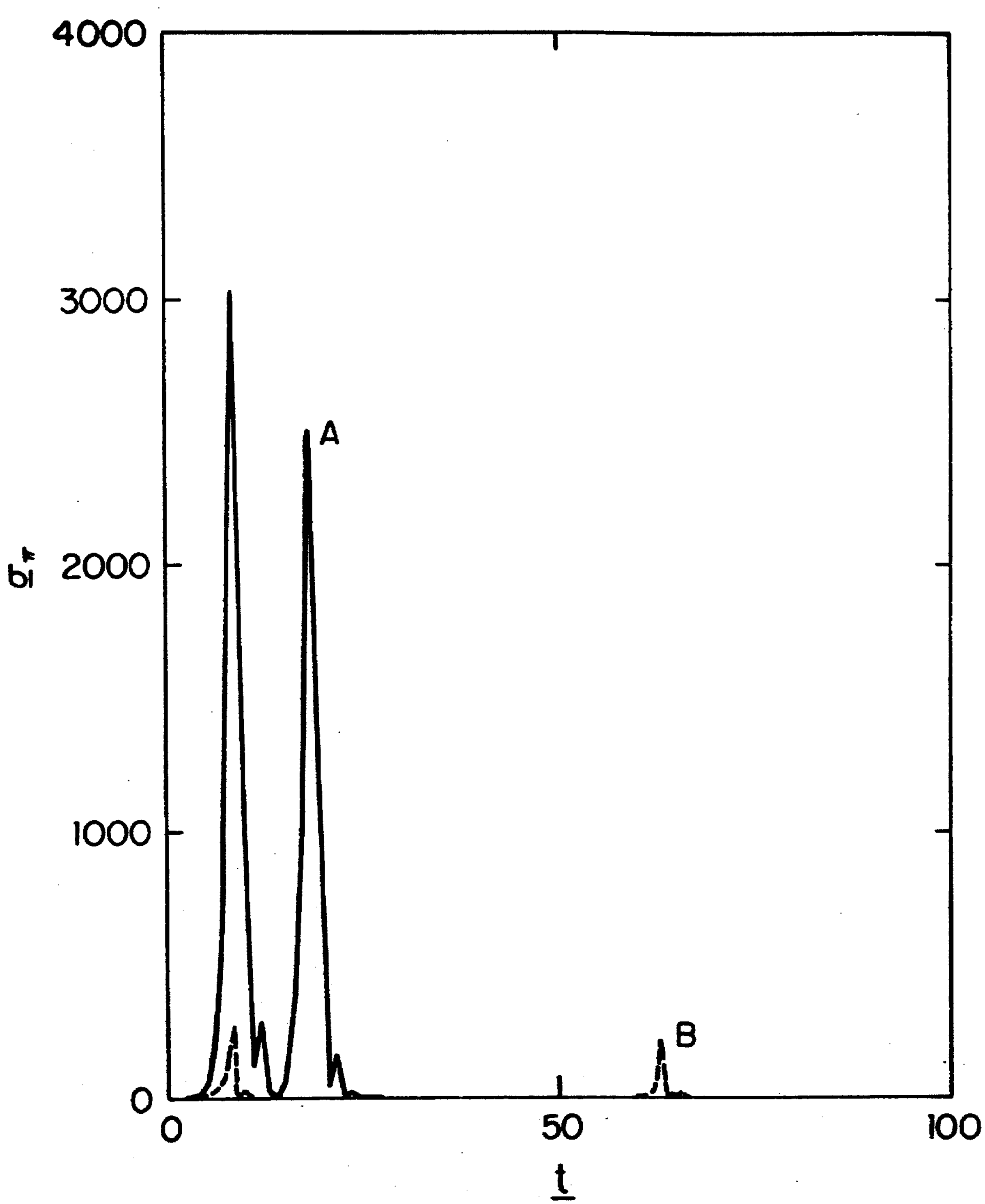

Fig.7